Premises as Resources

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Abstract

Premises as resources; Weakening and Contraction; central notions of relevance logic and linear logic; linear connectives.

1 Introduction

These are the notes for the second of a series of brief, informal talks on resource logics, functional programming, and sundry related matters, held under the auspices of Scheme UK.

In this talk, we will introduce the notion of premises as resources, and consider the role played by the structural rules of Weakening and Contraction in a logical system. We will then discuss the motivating ideas of relevance logic and linear logic, and finish by introducing some linear connectives.

As we will no longer be concerned with the rule of Exchange, we will henceforth treat $\Gamma, \Delta, \Sigma, \ldots$ as ranging over *multisets* of formulae —i.e., equivalence classes of sequences modulo ordering.

2 Premises as Resources

Both classical and intuitionistic logic are concerned with mathematical truth. Mathematical truth is an unlimited resource: it does not get 'used up'. If we use a fact to prove another fact, the first fact is still available, to be used again if we so desire: premises in classical and intuitionistic logic have the property of *unlimited reusability*. Moreover, there is no obligation to use all the facts at our disposal:

we can include premises in our derivations that do not contribute to the conclusion we ultimately derive, and there is no associated cost in doing so. That is, mathematical truth is cheap to the point of being *free*.

The upshot is that there are no budgetary restrictions on how premises are deployed: we don't need to use all our premises, nor do we need to worry about how many times we use each one. This is reflected in the *unrestricted* validity of the structural rules of *Weakening* and *Contraction* (repeated here for convenience):

Definition 1 (Weakening)

$$\frac{\Gamma \vdash \beta}{\Gamma, \alpha \vdash \beta}$$

Definition 2 (Contraction)

$$\frac{\Gamma, \alpha, \alpha \vdash \beta}{\Gamma, \alpha \vdash \beta}$$

This may be fine for mathematics, but there are contexts for which the view of truth as an unlimited and free resource is inappropriate; for example, classical logic is poorly suited to reasoning about contexts in which *state* plays a role, or where the truth of a premise involves some *cost*.

3 Relevance Logic

Suppose that, instead of getting our premises 'for free', we had to purchase them from some hypothetical "Premise Bank" (however, let us retain the assumption that premises are reusable). Then, in order to minimise our expenditure, we would want to ensure that we buy only those premises that we really need—i.e., only those premises that are *relevant* to the desired conclusion. *Relevance* logics are a family of systems that (among other things) minimise expenditure on premises, and in which the structural rule of *Weakening* is therefore not valid (in its unrestricted form). That is, we might have $\Gamma \vdash_R \beta$ without also having $\Gamma, \alpha \vdash_R \beta$ (the new material might not be relevant to the consequent).

Example 1 The following (classically valid) derivation is *invalid* from a relevantist perspective (note the explicit use of Weakening in the second step):

$$\dfrac{\dfrac{\dfrac{\overline{\beta \vdash \beta}\; Id}{\beta, \alpha \vdash \beta}\; Weakening}{\beta \vdash (\alpha \to \beta)}\; {\to}I}{\vdash \beta \to (\alpha \to \beta)}\; {\to}I$$

Example 2 Explicit uses of Weakening are not the only way in which irrelevant premises may sneak in. In the following derivation, it is the rule of $\wedge$ **I** that allows the introduction of the irrelevant premise p:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{p \vdash p}\,^{Id} \qquad \overline{q \vdash q}\,^{Id}}{p, q \vdash (p \wedge q)}\; \wedge I}{p, q \vdash q}\; \wedge E}{p \vdash (q \to q)}\; {\to}I}{\vdash p \to (q \to q)}\; {\to}I$$

One solution is to restrict the rule of $\wedge$ **I** to only allow conjunction of sequents which share identical premises:

Definition 3 ($\wedge$ I)

$$\dfrac{\Gamma \vdash_R \alpha \qquad \Gamma \vdash_R \beta}{\Gamma \vdash_R (\alpha \wedge \beta)}\; \wedge I$$

It is easy to check that this has the effect of blocking the previous derivation.

4 Linear Logic

Let us now add the further restriction that premises, instead of being infinitely reusable, get *consumed* in the course of a derivation—much like money, petrol, or bullets are in the course of shopping, driving, or shooting (respectively). *Linear* logics are logical systems that minimise expenditure on *consumable* premises: in such a logic, not only must premises be relevant, but they can be used only once. The effect of the additional restriction is that we might have $\Gamma, \alpha, \alpha \vdash \beta$ (which says that we need to use α twice to get β), but we might not have $\Gamma, \alpha \vdash \beta$ (which says that we only need to use α once to get β). Accordingly, *Contraction* is not valid (in its unrestricted form). A linear logic is thus one that is sensitive to both *cost* (premises are not free) and *state* (the status of a premise may change over the course of a derivation). Linear logics therefore provide a systematic means of reasoning about *processes*, including computational processes, which incur costs and involve changes of state.

Definition 4 (Linear Sequent) *A* linear sequent *has the form* $\Gamma \Vdash \varphi$*, and is read as 'If we had resources* Γ*, we could achieve the goal* φ*'.*

4.1 Some Linear Connectives

4.1.1 Linear Implication

We will read the linear implication $(\alpha \multimap \beta)$ as 'Consuming α yields β'.

Definition 5 ($\multimap$ I)

$$\frac{\Gamma, \alpha \Vdash \beta}{\Gamma \Vdash (\alpha \multimap \beta)}$$

Definition 6 ($\multimap$ E)

$$\frac{\Gamma \Vdash (\alpha \multimap \beta) \qquad \Delta \Vdash \alpha}{\Gamma, \Delta \Vdash \beta}$$

4.1.2 Simultaneous Conjunction

We write $(\alpha \otimes \beta)$ if our resources allow us to achieve α and β in the *same* state.

Definition 7 ($\otimes$ I)

$$\frac{\Gamma \Vdash \alpha \qquad \Delta \Vdash \beta}{\Gamma, \Delta \Vdash (\alpha \otimes \beta)}$$

Definition 8 ($\otimes$ E)

$$\frac{\Gamma \Vdash (\alpha \otimes \beta) \qquad \Delta, \alpha, \beta \Vdash \theta}{\Gamma, \Delta \Vdash \theta}$$

4.2 Alternative Conjunction

We write $(\alpha \mathbin{\&} \beta)$ if our resources allow us to achieve our choice of α and β, but not both together. For example, if you have one bullet, you can shoot the sheriff or you can shoot his deputy, but not both at the same time.[1]

Definition 9 (& I)

$$\frac{\Gamma \Vdash \alpha \qquad \Gamma \Vdash \beta}{\Gamma \Vdash (\alpha \mathbin{\&} \beta)}$$

Definition 10 (& E)

1.

$$\frac{\Gamma \Vdash (\alpha \mathbin{\&} \beta)}{\Gamma \Vdash \alpha}$$

2.

$$\frac{\Gamma \Vdash (\alpha \mathbin{\&} \beta)}{\Gamma \Vdash \beta}$$

5 Conclusion

The motivating insight of linear logic is the view of premises as *precious and consumable* resources. Everything else follows systematically from this idea.

In the next talk, we will present linear logic in a more systematic fashion, and relate it to the isomorphism between proofs and programs that we introduced in the first talk.

[1] Barring bizarre and exceptional circumstances.